

Universality of equilibrium one-dimensional transport from gauge invariance

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In this letter we address the question how interactions affect the DC conductance of a one-dimensional electron system not necessarily adequately described by the Luttinger model. Using a Laughlin type argument, we show that gauge invariance protects the universal value of the conductance of e^2/h per channel per spin orientation if the system possesses two conserved charges conjugate to the chemical potentials of the external reservoirs.

The DC conductance of a clean one-dimensional electron system described by the Luttinger model is given by the universal value of e^2/h per channel and per spin orientation [1]. In this letter we address the question whether the conductance can deviate from this universal value in situations where the Luttinger model may not be applicable.

Such a situation may be met, for example, if one considers a system with a strong bias, so that the Luttinger fixed point may no longer describe the state of the system. In principle, one should then take into account an infinite number of irrelevant operators which were suppressed at the fixed point. One might expect that this would lead to a nonlinear dependence of the current on the voltage drop and to a differential conductance depending on the bias.

Another possibility is the existence of fixed points of one-dimensional electron systems different from the Luttinger models. The latter arise as a result of a perturbative RG analysis for short range repulsive potentials. One cannot exclude the possibility that there exist other conformal theories describing one-dimensional electrons.

Our analysis is based on a Laughlin type argument [2] and relate the quantization of the conductance in a one-dimensional electron system to gauge invariance. The only conditions imposed on the dynamics of the system are the existence of conserved charges conjugate to the chemical potentials of the reservoirs and the absence of impurity backscattering. Under these assumptions, we show that the conductance is equal to e^2/h per channel and per spin orientation.

Recently, experiments were done measuring the conductance in very pure quantum wires [3]. At small bias, experimental data show that the value of the conductance is somewhat smaller than the universal value: $G \simeq ng(T)2e^2/h$, $n = 0, 1, 2, \dots$, where $g(T) < 1$ is a factor depending on the temperature T . The factor $g(T)$ can be interpreted as arising from a renormalization of the voltage drop and must be explained in terms of the physics of the leads and of the junctions between the wire and the leads.

To describe the conductance of a one-dimensional system we use an extension of the Landauer-Büttiker approach [4] to the case of interacting systems [5]. We con-

sider a one-dimensional system put into contact with two external reservoirs with different chemical potentials, μ_L and μ_R . We assume that there are two conserved charges, Q_L and Q_R , conjugate to μ_L and μ_R , which commute with the Hamiltonian \mathcal{H} of the system and with each other:

$$[\mathcal{H}, Q_L] = 0, [\mathcal{H}, Q_R] = 0, [Q_L, Q_R] = 0. \quad (1)$$

Then the thermal state of the system connected to the external reservoirs is given by the density matrix

$$\Sigma_\mu = \exp(-\beta(\mathcal{H} + \mu_L Q_L + \mu_R Q_R)) \quad (2)$$

and its transport properties are described by *equilibrium* statistical mechanics.

A special property of one-dimensional electric transport is that the continuity equation for the electric current

$$\partial_t \rho + \partial_x j = 0 \quad (3)$$

is solved by introducing a scalar operator ϕ

$$\rho = e\partial_x \phi, \quad j = -e\partial_t \phi. \quad (4)$$

Here e is the elementary electric charge. This enables one to derive a formula which is a one-dimensional DC analogue of the Kubo formula. In the equilibrium state characterized by the chemical potentials μ_L and μ_R the expectation value of the current is given by

$$\begin{aligned} \langle j(x) \rangle_\mu &= -e\langle \partial_t \phi(x) \rangle_\mu = \frac{i}{\hbar} \langle [\mathcal{H}, \phi(x)] \rangle_\mu = \\ &= \frac{i}{\hbar} (\text{Tr} \Sigma_\mu)^{-1} \text{Tr} (\Sigma_\mu [\mathcal{H}, \phi(x)]) = \\ &= -\frac{i}{\hbar} \langle [\mu_L Q_L + \mu_R Q_R, \phi(x)] \rangle_\mu. \end{aligned} \quad (5)$$

Formula (5) expresses the current in terms of the commutation relations of the conserved charges with the bosonic field ϕ providing a solution of the continuity equation.

Before applying formula (5) to a general situation, we consider two simple examples. We start with a zero-coupling Luttinger model described by the Lagrangian

$$\mathcal{L} = i\hbar\psi_L^*(\partial_t - v_F\partial_x)\psi_L + i\hbar\psi_R^*(\partial_t + v_F\partial_x)\psi_R. \quad (6)$$

The corresponding Hamiltonian looks as follows

$$\mathcal{H}_0 = i\hbar v_F \int dx (\psi_L^* \partial_x \psi_L - \psi_R^* \partial_x \psi_R). \quad (7)$$

Standard bosonization formulas,

$$\begin{aligned} \psi_L^* \psi_L &= n_L, \quad \psi_R^* \psi_R = n_R; \\ i\hbar \psi_L^* \partial_x \psi_L &= \frac{\hbar}{2} n_L^2, \\ -i\hbar \psi_R^* \partial_x \psi_R &= \frac{\hbar}{2} n_R^2 \end{aligned} \quad (8)$$

yield the following expression for the bosonized Hamiltonian

$$\mathcal{H}_0 = \frac{\hbar v_F}{2} \int dx (n_L^2 + n_R^2). \quad (9)$$

Coupling the system to two external reservoirs which feed left- and right-moving electrons into the system can be described by adding to the Hamiltonian an extra term which includes the chemical potentials μ_L and μ_R of the reservoirs:

$$\mathcal{H}_\mu = \mathcal{H}_0 + \mu_L Q_L + \mu_R Q_R, \quad (10)$$

where Q_L and Q_R are conserved particle numbers of left- and right-moving particles:

$$Q_L = \int dx n_L, \quad Q_R = \int dx n_R. \quad (11)$$

Minimizing the expectation value of the Hamiltonian (10) gives

$$n_L = -\frac{\mu_L}{\hbar v_F}, \quad n_R = -\frac{\mu_R}{\hbar v_F}. \quad (12)$$

The total current $j = ev_F(n_R - n_L)$ is given by the Landauer-Büttiker formula

$$j = \frac{e(\mu_L - \mu_R)}{\hbar} = \frac{e^2}{\hbar} V, \quad (13)$$

where V is the voltage drop $eV = \mu_L - \mu_R$.

Next, we study the effect of adding an interaction term described by the following irrelevant operator

$$\Delta \mathcal{H} = \varepsilon \int dx (\psi_L^* \partial_x \psi_L)(\psi_R \partial_x \psi_R). \quad (14)$$

The bosonized interacting Hamiltonian is given by

$$\mathcal{H}_{int} = \mathcal{H}_\mu + \varepsilon \pi^2 \int dx n_L^2 n_R^2. \quad (15)$$

The expectation value of \mathcal{H}_{int} achieves its minimum when

$$\begin{aligned} n_L + \frac{2\pi^2 \varepsilon}{\hbar v_F} n_L n_R^2 &= -\frac{\mu_L}{\hbar v_F}, \\ n_R + \frac{2\pi^2 \varepsilon}{\hbar v_F} n_R n_L^2 &= -\frac{\mu_R}{\hbar v_F}. \end{aligned} \quad (16)$$

Before computing the current in the interacting system we should check whether the interaction can possibly change the definition of the current in terms of j_L and j_R . The current is given by the general formula

$$j(x) = -\left. \frac{\delta \mathcal{H}}{\delta a(x)} \right|_{a=0}, \quad (17)$$

where a is the spatial component of the vector potential. The latter enters the Hamiltonian through the covariant derivatives $iD_x = i\hbar \partial_x + ea$. For instance, the non-interacting Hamiltonian acquires the form

$$\mathcal{H}_0(a) = \mathcal{H}_0 + ev_F \int dx a(n_L - n_R). \quad (18)$$

This implies that $j = ev_F(n_L - n_R)$, as we stated before.

By applying the same procedure to the interacting Hamiltonian we easily arrive at

$$\begin{aligned} \mathcal{H}_{int}(a) &= \mathcal{H}_0(a) + \\ &+ \varepsilon \pi^2 \int dx (n_L^2 + 2\frac{e}{\hbar} a n_L)(n_R^2 - 2\frac{e}{\hbar} a n_R). \end{aligned} \quad (19)$$

Therefore, the current gets an extra contribution from the interaction term:

$$j_{int} = ev_F(n_R - n_L) - \varepsilon \frac{2e}{\hbar} n_L n_R (n_R - n_L). \quad (20)$$

In the ground state of the interacting Hamiltonian (16), the expectation value of this expression is equal to $e(\mu_L - \mu_R)/\hbar$. Thus, the redefinition of the current exactly compensates the extra contribution caused by the interaction.

Another example of an interacting electronic system is an interacting Luttinger model which is obtained from the non-interacting one by adding a marginal perturbation

$$\Delta \mathcal{H}_{Lut} = \hbar \varepsilon v_F \int dx (\psi_L^* \psi_L)(\psi_R^* \psi_R). \quad (21)$$

This perturbation changes some of the bosonization rules (8):

$$\begin{aligned} i\hbar \psi_L^* \partial_x \psi_L &= \frac{\hbar(1+\varepsilon)}{2} n_L^2, \\ -i\hbar \psi_R^* \partial_x \psi_R &= \frac{\hbar(1+\varepsilon)}{2} n_R^2. \end{aligned} \quad (22)$$

The bosonized Hamiltonian is given by

$$\begin{aligned} \mathcal{H}_{Lut}(a) &= \hbar v_F \int dx \left(\frac{1+\varepsilon}{2} (n_L^2 + n_R^2) + \varepsilon n_L n_R \right) + \\ &+ ev_F \int dx a(n_L - n_R) + \mu_L Q_L + \mu_R Q_R. \end{aligned} \quad (23)$$

The minimum at $a = 0$ is achieved when

$$(1 + \varepsilon)n_L + \varepsilon n_R = -\frac{\mu_L}{\hbar v_F};$$

$$(1 + \varepsilon)n_R + \varepsilon n_L = -\frac{\mu_R}{\hbar v_F}. \quad (24)$$

The current is given by the same universal formula (13).

We have seen that the interaction does not affect the conductance for two different types of interaction. For the marginal interaction, the bosonization formulas read just so as to compensate the effect of interactions on the current. For irrelevant interactions considered above, the conductance is not changed because the expression for the current density in terms of n_L and n_R changes.

Next, we turn to a proof of the general statement that formula (13) does not depend on the concrete dynamics of the one-dimensional electron system, as long as the reservoirs are coupled to the system via conserved particle numbers Q_L and Q_R .

Independently of the structure of interactions, the charge density has the form

$$\rho = e(\psi_L^* \psi_L + \psi_R^* \psi_R). \quad (25)$$

We recall that the expression for the electric current is sensitive to the particular form of interactions.

It is convenient to introduce one-dimensional bosonization formulas for Fermi fields ψ_L and ψ_R :

$$\psi_L^* = e^{2\pi i \phi_L}, \quad \psi_L = e^{-2\pi i \phi_L};$$

$$\psi_R^* = e^{-2\pi i \phi_R}, \quad \psi_R = e^{2\pi i \phi_R}. \quad (26)$$

The bosonic fields ϕ_L and ϕ_R satisfy the following commutation relations

$$[\phi_L(x), \phi_L(y)] = \frac{i}{4\pi} \varepsilon(x - y),$$

$$[\phi_R(x), \phi_R(y)] = -\frac{i}{4\pi} \varepsilon(x - y), \quad (27)$$

$$[\phi_L(x), \phi_R(y)] = \frac{i}{4\pi},$$

where $\varepsilon(x - y) = 1, x > y; \varepsilon(x - y) = -1, x < y$. The densities of left- and right-moving particles acquire the form

$$n_L = \partial_x \phi_L, \quad n_R = \partial_x \phi_R. \quad (28)$$

The conserved charges Q_L and Q_R have the following commutation relations with the bosonic fields:

$$[Q_L, \phi_L(x)] = \frac{i}{2\pi}, \quad [Q_R, \phi_R(y)] = -\frac{i}{2\pi}. \quad (29)$$

One can introduce a bosonic field $\phi(x) = \phi_L(x) + \phi_R(x)$ and rewrite the charge density as follows

$$\rho = e(\partial_x \phi_L + \partial_x \phi_R) = e \partial_x \phi. \quad (30)$$

We note that all the commutation relations and bosonization rules listed above depend only on the kinematics of Fermi fields and are entirely independent of the dynamics

of the model we consider. The only important assumption which we make is commutativity of the charges Q_L and Q_R with the Hamiltonian of the interacting system.

In order to use the definition of the electric current, we should specify how the vector potential a enters the Hamiltonian. This is accomplished by a simple substitution

$$\phi_L(x) \rightarrow \phi_L^a(x) = \phi_L(x) + \frac{e}{\hbar} \int_{-\infty}^x dy a(y);$$

$$\phi_R(x) \rightarrow \phi_R^a(x) = \phi_R(x) - \frac{e}{\hbar} \int_{-\infty}^x dy a(y). \quad (31)$$

This rule is equivalent to replacing partial derivatives ∂_x of fermionic operators by covariant derivatives. One can easily check it using (26). Now we are ready to compute the electric current:

$$j(x) = - \left. \frac{\delta \mathcal{H}(\phi_L^a, \phi_R^a)}{\delta a(x)} \right|_{a=0} =$$

$$= i \frac{e}{\hbar} [\mathcal{H}(\phi_L, \phi_R), \phi_L(x) + \phi_R(x)] = -e \partial_t \phi. \quad (32)$$

Here we used the Leibniz rule for derivatives and commutators and the following formulas obtained by combining (27) and (31)

$$\frac{\delta \phi_L^a(x)}{\delta a(y)} = -i \frac{e}{\hbar} [\phi_L^a(x), \phi_L(y) + \phi_R(y)];$$

$$\frac{\delta \phi_R^a(x)}{\delta a(y)} = -i \frac{e}{\hbar} [\phi_R^a(x), \phi_L(y) + \phi_R(y)]. \quad (33)$$

The electric current in bosonized form automatically fulfills the continuity equation:

$$\partial_t \rho + \partial_x j = e \partial_t \partial_x \phi - e \partial_x \partial_t \phi = 0. \quad (34)$$

Formulas (29) and (32) can now be used to compute the conductance from (5):

$$\langle j(x) \rangle_\mu = -i \frac{e}{\hbar} \langle [\mu_L Q_L + \mu_R Q_R, \phi(x)] \rangle_\mu =$$

$$= \frac{e}{\hbar} (\mu_L - \mu_R). \quad (35)$$

This confirms the universal conductance formula (13) in the general situation.

This result can be understood by using a simple argument similar to the one suggested by Laughlin [2] as an explanation of the integer Quantum Hall effect. The most general form of the bosonized Hamiltonian which commutes with the conserved charges Q_L and Q_R is $\mathcal{H} = \mathcal{H}(\rho, \eta, a)$, where ρ is the charge density and $\eta = (n_L - n_R)$ is the difference of the particle densities of left- and right-movers. Let us for simplicity assume that $\mu_L + \mu_R = 0$.

The energy minimum of \mathcal{H}_μ is achieved if

$$\left. \frac{\delta \mathcal{H}}{\delta \eta(x)} \right|_\rho = \frac{1}{2} (\mu_R - \mu_L). \quad (36)$$

The current in the system is defined by

$$j(x) = -\frac{\delta\mathcal{H}}{\delta a(x)}\Big|_{a=0}. \quad (37)$$

In order to relate the left hand side of (36) to the right hand side of (37) one applies Laughlin's argument. More precisely, we assume that our system is put on a ring which encircles a magnetic flux Φ . Let us adiabatically change Φ by one flux quantum

$$\Phi \rightarrow \Phi + \frac{h}{e}. \quad (38)$$

If the system was in the ground state at the beginning of the process it will be in a new ground state at the end. As the change of Φ by a flux quantum can be compensated by a gauge transformation, this will be a new ground state of the *same* Hamiltonian. Formulas (31) imply that in this process $Q_L = \int dx \partial_x \phi_L$ increases by 1 whereas $Q_R = \int dx \partial_x \phi_R$ decreases by 1. Thus, one particle is transferred from the left-moving channel to the right-moving one. This implies

$$\frac{e}{h} da|_\rho = \frac{1}{2} d\eta|_\rho. \quad (39)$$

By combining (39) with (36) and (37) one obtains the universal result

$$j = -\frac{\delta\mathcal{H}}{\delta a(x)}\Big|_{a=0} = -\frac{2e}{h} \frac{\delta\mathcal{H}}{\delta\eta}\Big|_\rho = \frac{e}{h}(\mu_L - \mu_R). \quad (40)$$

These considerations are not applicable if the interacting Hamiltonian includes operators of the type $\cos(2\pi n\phi)$ responsible for backscattering [6]. Although, such operators may be irrelevant, they break the symmetry generated by the charges Q_L and Q_R , and hence formula (2) for the thermal state is no longer applicable.

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